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Input states for quantum gates

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We examine three possible implementations of nondeterministic linear optical controlled NOT gates with a view to an in-principle demonstration in the near future. To this end we consider demonstrating the gates using currently available sources, such as spontaneous parametric down conversion and coherent states, and current detectors only able to distinguish between zero and many photons. The demonstration is possible in the coincidence basis and the errors introduced by the nonoptimal input states and detectors are analyzed.

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I. INTRODUCTION

Optics is a natural candidate for implementing a variety of quantum-information protocols. Photons make beguiling qubits: at optical frequencies the qubits are largely decoupled from the environment and so experience little decoherence, and single-qubit gates are easily realized via passive optical elements. Some protocols, notably quantum computation, also require two-qubit gates. Until recently this was regarded as optically infeasible, since the required nonlinear interaction is much greater than that available with extant materials. However, it is now widely recognized that the necessary nonlinearity can be realized nondeterministically via measurement, and that deterministic gates can be achieved by combining such nondeterministic gates and teleportation [1].

There are a number of proposals for implementing a nondeterministic controlled NOT (CNOT) gate with linear optics and photodetectors [1–6]. The proposals require deterministic, or heralded, single-photon sources and/or *selective* detectors that can distinguish with very high efficiency between zero, one, and multiple photons. Current commercial optical sources and detectors fall well short of these capabilities. Although there are a number of active research programs aimed at producing both efficient selective detectors [7,8] and deterministic photon sources [9–11], nonselective avalanche photodiodes, spontaneous parametric down-conversion (SPDC) and coherent states remain the best accessible laboratory options. While we could side step the single-photon source problem by using an SPDC source conditioned on the detection of a photon in one arm if we had selective detectors, demonstrating a four-photon CNOT gate without quantum memory would be frustratingly slow.

In this paper we examine three proposals, which allow a CNOT operation to be implemented nondestructively on the control and target modes, to ascertain under what conditions it is possible to demonstrate and characterize the gates operation using SPDC sources, coherent states, and *nonselective* detectors (detectors only able to resolve zero and multiple photons). The aim is to identify a scheme that allows a scalable CNOT implementation to be initially examined with current sources and detectors, and into which we can easily incorporate single-photon sources and selective detectors as they become available.

Typically the gates involve four photons with the qubit states encoded in the polarization state of the control and target modes c and t, and the CNOT operation is implemented with the aid of some ancillary modes a, b, etc. We will consider starting with the control and target modes each in a general superposition (we could also consider initially entangled states though these may be more difficult experimentally),

$$|\psi_{\rm in}\rangle_{ct}=(A_h\hat{\boldsymbol{c}}_h^\dagger+A_v\hat{\boldsymbol{c}}_v^\dagger)(B_h\hat{\boldsymbol{t}}_h^\dagger+B_v\hat{\boldsymbol{t}}v^\dagger)|\mathbf{0}\rangle, \tag{1}$$

with $|A_h|^2+|A_v|^2=|B_h|^2+|B_v|^2=1$, and where $\hat{\boldsymbol{c}}_{h,v}^\dagger$ and $\hat{\boldsymbol{t}}_{h,v}^\dagger$ are bosonic creation operators for modes $c_{h,v}$ and $t_{h,v}$, respectively. In the interest of brevity we will use the notation above where we write the state in terms of creation operators acting on the vacuum state.

The modes are first manipulated with a linear optics network $U_{\rm CNOT}$ comprising beam splitters, phase shifters, wave plates, and polarizing beam splitters. Finally, the gate is conditioned on detecting the ancillary modes in some appropriate state, leaving the state of the control and target modes as if a CNOT gate had been applied.

The key simplification for our purposes is to detect in the "coincidence basis"—where we detect the output of the ancillary modes and also of the target and control modes and postselect out those events that do not simultaneously register a photon in all four modes. The advantage of this configuration is that now we can use *nonselective* detectors, since if we get a "click" on all four detectors we have accounted for all the photons in the system. This is a much less stringent requirement on the detectors, and, in particular, can be fulfilled by existing avalanche photodiodes. We model the nonselective detectors with a positive-operator-valued measure (POVM), with the POVM elements associated with detecting no photons or photons (one or more) simply being $\Pi_0=|0\rangle\langle 0|$ and $\Pi_m=\Sigma_{n=1}^{\infty}|n\rangle\langle n|$, respectively.

The output state of a type-I SPDC can be described as

$$|\lambda\rangle=\mathcal{M}_\lambda\sum_{\substack{n=0\\ \text{(even)}}}^{\infty}\frac{(\lambda\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{b}}^\dagger)^{n/2}}{(n/2)!}|\mathbf{0}\rangle, \tag{2}$$

where $\mathcal{M}_\lambda=(1-\lambda^2)^{-1/2}$ and the sum is over even n where n is the number of photons in each term.

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Now suppose that our input state to the optical circuit is some initial pure state $|\psi_{\rm in}\rangle$, and that after passing through the linear optical elements we are left in the state $|\psi_{\rm out}\rangle = U_{\rm CNOT}|\psi_{\rm in}\rangle$. The probability that we get a count simultaneously in modes c, t, a, and b with nonselective detectors is

$$P=\langle\psi_{\rm out}|\Pi_m^{(c)}\otimes\Pi_m^{(t)}\otimes\Pi_m^{(a)}\otimes\Pi_m^{(b)}|\psi_{\rm out}\rangle. \quad (3)$$

For the ideal case where we had single-photon inputs to the gate, we will label this probability as P_1. We can now introduce the "single-photon visibility" as a figure of merit for how close the gate operates to the ideal

$$\mathcal{V}=\frac{1}{2}\left(\frac{s-e}{s+e}+1\right), \quad (4)$$

where s is the product of the probability of obtaining the single-photon terms from the source, with P_1 the probability of the gate functioning. The "error" $e=\max(s-P)$, where P is the actual probability of obtaining a count on the detectors. The maximization is over all qubit input states to the gate. Hence, if the error totally dominates the visibility is close to 0, if the noise is small the visibility is close to 1. As a guide, a visibility of 0.8 corresponds to an error a quarter of the size of the single-photon "signal" s.

II. SIMPLIFIED KNILL-LAFLAMME-MILBURN CNOT GATE

In the originally proposed nondeterministic CNOT gate [1], the nonlinear sign shift elements were interferometric: these elements can be replaced by sequential beam splitters to make a simplified CNOT gate [2], one example of which is (refer to Fig. 4 of Ref. [2])

$$\hat{\boldsymbol{U}}_{\rm SKLM}=\hat{\boldsymbol{B}}_{t_ht_v}\left(\frac{\pi}{4}\right)\hat{\boldsymbol{B}}_{c_vt_h}\left(\frac{\pi}{4}\right)\hat{\boldsymbol{B}}_{bt_h}(\theta_2)\hat{\boldsymbol{B}}_{ac_v}(\theta_2)\hat{\boldsymbol{B}}_{c_vt_h}\left(\frac{\pi}{4}\right)$$
$$\times\hat{\boldsymbol{B}}_{t_hv_2}(\theta_1)\hat{\boldsymbol{B}}_{t_ht_v}\left(\frac{\pi}{4}\right)\hat{\boldsymbol{B}}_{v_1c_v}(\theta_1), \quad (5)$$

where $\hat{\boldsymbol{B}}_{ab}$ represents a beam splitter with the following actions: $\hat{\boldsymbol{B}}_{ab}(\theta)\hat{\boldsymbol{a}}\hat{\boldsymbol{B}}_{ab}^\dagger(\theta)=\hat{\boldsymbol{a}}\cos\theta+\hat{\boldsymbol{b}}\sin\theta$, $\hat{\boldsymbol{B}}_{ab}(\theta)\hat{\boldsymbol{b}}\hat{\boldsymbol{B}}_{ab}^\dagger(\theta)=\hat{\boldsymbol{a}}\sin\theta-\hat{\boldsymbol{b}}\cos\theta$, and $\cos^2\theta$ is the reflectivity. The angle choices for the gate are given by $\theta_1=\cos^{-1}\sqrt{5-3\sqrt{2}}$ and $\theta_2=\cos^{-1}\sqrt{(3-\sqrt{2})/7}$; c and t are the control and target modes and a, b, v_1, and v_2 are independent ancillary modes. The gate is conditioned on detecting a single-photon in the modes a and b and detecting no photons in the modes v_1 and v_2.

Consider the case where both the control, target, and ancillary photons are supplied by two independent SPDC sources. The input state is $|\lambda\rangle_{ct}|\epsilon\rangle_{ab}$, which can be written as a sum over total photon number

$$|\phi_{\rm in}\rangle=\mathcal{M}_\lambda\mathcal{M}_\epsilon\sum_{\substack{n=0\\(\text{even})}}^{\infty}\hat{\boldsymbol{Q}}_n|\boldsymbol{0}\rangle,$$

$$\hat{\boldsymbol{Q}}_n=\sum_{m=0}^{n/2}\frac{\epsilon^m\lambda^{n/2-m}}{m!\left(\frac{n}{2}-m\right)!}(\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{b}}^\dagger)^m(\hat{\boldsymbol{c}}^\dagger\hat{\boldsymbol{t}}^\dagger)^{n/2-m}. \quad (6)$$

The control and target horizontal and vertical polarization modes are then each mixed on a beam splitter so that we achieve the input state (1) for those modes.

Since we are postselecting on getting a "click" at four detectors, then the terms with $n<4$ will always get postselected out. Similarly, the terms with $n>4$ will get postselected out if we used selective detectors, otherwise they represent error terms. In the latter case, so long as $\epsilon,\lambda\ll1$ these terms will be small. For the case where $n=4$, three input terms contribute

$$|\psi_{\rm in}^{(4)}\rangle=\left(\lambda\epsilon\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{b}}^\dagger\hat{\boldsymbol{c}}^\dagger\hat{\boldsymbol{t}}^\dagger+\frac{\lambda^2}{2!}\hat{\boldsymbol{c}}^{\dagger2}\hat{\boldsymbol{t}}^{\dagger2}+\frac{\epsilon^2}{2!}\hat{\boldsymbol{a}}^{\dagger2}\hat{\boldsymbol{b}}^{\dagger2}\right)|\boldsymbol{0}\rangle. \quad (7)$$

While the first of these terms is equivalent to having four initial Fock states, the remaining two terms have the possibility of surviving the postselection criteria and skewing the statistics observed. Fortunately these last two terms lead to output terms, which *all* get postselected out in the coincidence basis (e.g., two photons in the control mode). This means that with selective detectors we could in principle postselect out all terms that do not correspond to single-photon inputs from the output statistics. With nonselective detectors the error terms will scale at least as λ^3 in amplitude (due to the $n>4$ terms), so the figure of merit will scale with λ (taking $\epsilon=\lambda$) as $\mathcal{V}\sim1/(1+\lambda^2)$ and λ is typically very small.

Now consider the situation where a SPDC supplies the two photons for the control and target modes, and weak coherent states are used for the ancillary modes. The input state is then $|\phi_{\rm in}\rangle=|\lambda,\alpha,\beta\rangle$, where $\hat{\boldsymbol{a}}^\dagger$ and $\hat{\boldsymbol{b}}^\dagger$ will be the creation operators for the coherent states. After rearranging the state as a primary sum over photon number we get

$$|\phi_{\rm in}\rangle=\sum_{n=0}^{\infty}\sum_{\substack{p=0\\(\text{even})}}^{n}\sum_{q=0}^{n-p}\frac{(\lambda\hat{\boldsymbol{c}}^\dagger\hat{\boldsymbol{t}}^\dagger)^{p/2}}{(p/2)!}\frac{(\alpha\hat{\boldsymbol{a}}^\dagger)^q}{q!}\frac{(\beta\hat{\boldsymbol{b}}^\dagger)^{n-p-q}}{(n-p-q)!}|\boldsymbol{0}\rangle. \quad (8)$$

Again, terms with $n<4$ will get postselected out and terms with $n>4$ will be weak error terms. The extra freedom from two independent coherent states means that now there will be nine terms with $n=4$ and only one of these is equivalent to using single-photon inputs.

Postselection removes the terms that arise from a coherent state in one of the modes supplying *all* four photons. By setting $\beta=i\alpha$ the two terms where a single coherent state supplies two photons and the SPDC supplies two will cancel each other due to the symmetry in the circuit. Finally, the term where the SPDC supplies all the photons is postselected out as before. This means that we will still get errors arising from the input terms:

$$\frac{i\alpha^4}{6}(\hat{\boldsymbol{a}}^{\dagger3}\hat{\boldsymbol{b}}^\dagger-\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{b}}^{\dagger3})|\boldsymbol{0}\rangle. \quad (9)$$

Note that these do not depend on the input state that is encoded on the control and target modes and by setting $\alpha \ll \lambda$ we can scale away these terms relative to the single-photon terms. Unfortunately this means that we cannot beat the photon collection rate that could be achieved using two independent SPDC sources.

It should be noted that all the observations made for the simplified Knill-Laflamme-Milburn (KLM) CNOT gate also hold for the full KLM CNOT gate in the coincidence basis. However, from the perspective of an initial demonstration of the gate the simplified version is more desirable. In the following two sections we will compare these results against two other implementations of optical CNOT gates.

III. ENTANGLED ANCILLA CNOT GATE

In a recent paper, Pittman, Jacobs, and Franson [6] proposed using entangled ancilla to further simplify implementating the CNOT operation. Consider that we have at our disposal an entangled state $|\phi\rangle = (\hat{\boldsymbol{a}}_h^\dagger \hat{\boldsymbol{b}}_h^\dagger + \hat{\boldsymbol{a}}_v^\dagger \hat{\boldsymbol{b}}_v^\dagger)/\sqrt{2}|\mathbf{0}\rangle$, then we can implement the CNOT operation between modes c and t by first applying the unitary (refer to Fig. 6 of Ref. [6])

$$\hat{\boldsymbol{U}}_{\text{ent}} = \hat{\boldsymbol{P}}_{bd}\hat{\boldsymbol{P}}_{ae}\hat{\boldsymbol{W}}_a\hat{\boldsymbol{W}}_t\hat{\boldsymbol{W}}_b\hat{\boldsymbol{P}}_{bt}\hat{\boldsymbol{W}}_t\hat{\boldsymbol{W}}_b\hat{\boldsymbol{P}}_{ac}, \tag{10}$$

where $\hat{\boldsymbol{W}}_a$ represents a half-wave plate on mode a; $\hat{\boldsymbol{P}}_{ab}$ is a polarizing beam splitter in modes a and b with the effect that $a_h \to a_h$, $b_h \to b_h$, $a_v \to b_v$, and $b_v \to a_v$; and d and e are extra output modes. Finally, the resulting state is then conditioned on detecting a single photon in modes a and b. The raw success probability of this gate is 1/16, which rises to 1/4 if fast feed forward and correction is used.

Consider that the entangled pair in modes a and b are provided by two type-I parametric down-converting crystals sandwiched together. We will fix the relative phase to get a particular Bell pair for the two-photon term

$$\begin{aligned} |\epsilon_2\rangle &= \mathcal{M}_\epsilon^2(|00\rangle + \epsilon|11\rangle + \cdots)(|00\rangle + \epsilon|11\rangle + \cdots) \\ &= \mathcal{M}_\epsilon^2[\cdots + \epsilon(|0011\rangle + |1100\rangle) + \cdots], \end{aligned} \tag{11}$$

where the modes are a_h, b_h, a_v, and b_v, respectively. Such sources have been previously built and provide a relatively bright source of polarization-entangled photons [12,13]. We can write this source succinctly as

$$|\epsilon_2\rangle = \mathcal{M}_\epsilon^2 \sum_{\substack{n=0 \\ \text{(even)}}}^{\infty} \hat{\boldsymbol{L}}_n |\mathbf{0}\rangle, \tag{12}$$

$$\hat{\boldsymbol{L}}_n = \sum_{m=0}^{n/2} \frac{\epsilon^{n/2}(\hat{\boldsymbol{a}}_h^\dagger \hat{\boldsymbol{b}}_h^\dagger)^m (\hat{\boldsymbol{a}}_v^\dagger \hat{\boldsymbol{b}}_v^\dagger)^{n/2-m}}{m!\left(\dfrac{n}{2}-m\right)!}. \tag{13}$$

With another independent SPDC source $|\lambda\rangle$, supplying the photons for the control and target modes, the input state becomes

$$|\phi_{\text{in}}\rangle \equiv \mathcal{M}_\epsilon^2 \mathcal{M}_\lambda \sum_{\substack{n=0 \\ \text{(even)}}}^{\infty} \sum_{\substack{q=0 \\ \text{(even)}}}^{n} \hat{\boldsymbol{L}}_q \frac{\lambda^{(n-q)/2}(\hat{\boldsymbol{c}}^\dagger \hat{\boldsymbol{t}}^\dagger)^{(n-q)/2}}{\left(\dfrac{n-q}{2}\right)!}, \tag{14}$$

where we will encode the qubits in the polarization state of the control and target modes, as in Eq. (1).

Again all terms with $n<4$ will get postselected out. There are six terms with $n=4$, of which two terms represent the single-photon input terms, the rest are error terms due to the sources. With nonselective detectors, terms with $n>4$ will also contribute to the error.

The four-photon terms in the output state that do not get postselected out are

$$\begin{aligned} |\text{out}\rangle = \frac{1}{2\sqrt{2}}\lambda\hat{\boldsymbol{a}}^\dagger\hat{\boldsymbol{b}}^\dagger(&A_vB_h\epsilon\hat{\boldsymbol{c}}_v^\dagger\hat{\boldsymbol{t}}_v^\dagger + A_vB_v\epsilon\hat{\boldsymbol{c}}_v^\dagger\hat{\boldsymbol{t}}_h^\dagger \\ &+ A_h[A_vB_h^2\lambda - A_vB_v^2\lambda + B_v\epsilon]\hat{\boldsymbol{c}}_h^\dagger\hat{\boldsymbol{t}}_v^\dagger \\ &+ A_h[A_vB_h^2\lambda - A_vB_v^2\lambda + B_h\epsilon]\hat{\boldsymbol{c}}_h^\dagger\hat{\boldsymbol{t}}_h^\dagger)|\mathbf{0}\rangle, \end{aligned} \tag{15}$$

and by making $\lambda \ll \epsilon$ we can recover the single-photon terms and the action of the CNOT gate with selective detectors. This, of course, means that the count rate with this gate would be considerably less than that with the simplified KLM gate. With nonselective detectors, if we make λ too small the error due to the six-photon input terms will dominate, so there is an optimum λ for a given ϵ, see Fig. 1(a).

There does not appear to be a way of using two coherent states to replace one of the SPDC sources. If we replace either the control or the target mode, then it is hard to see how the $|02\rangle$ and $|20\rangle$ terms could cancel as with the simplified KLM CNOT gate, since these terms will have factors that depend on the encoded qubit. Similarly, replacing the source of entangled photons would then mean we would have to entangle the single-photon components, which is difficult.

IV. KNILL CNOT GATE

A recent numerical search for optical gates by Knill yielded a CNOT gate [14], which operates with a probability of 2/27 and is described by the following unitary (refer to Fig. 1 of Ref. [14]):

$$\begin{aligned} \hat{\boldsymbol{U}}_{\text{Knill}} = &\hat{\boldsymbol{B}}_{t_vt_h}\left(\frac{\pi}{4}\right)\hat{\boldsymbol{B}}_{ab}(\theta_3)\hat{\boldsymbol{B}}_{c_vt_v}(\theta_2)\hat{\boldsymbol{B}}_{t_vb}(\theta_1)\hat{\boldsymbol{B}}_{c_va}(\theta_1) \\ &\times\hat{\boldsymbol{B}}_{t_vt_h}\left(\frac{\pi}{4}\right)\hat{\boldsymbol{F}}_a(\pi), \end{aligned} \tag{16}$$

where $\hat{\boldsymbol{F}}_a(\theta)$ is a phase shift of θ on mode a and the reflectivities are given by $\theta_1 = \cos^{-1}\sqrt{1/3}$, $\theta_2 = -\theta_1$, and $\theta_3 = \cos^{-1}\sqrt{1/2 + 1/\sqrt{6}}$. The gate requires two ancillary modes a and b initially in Fock states, to be finally detected also in single Fock states.

Consider the case where both the control, target, and ancillary photons are supplied by two independent SPDC sources. The input state is given by Eq. (6) with the usual

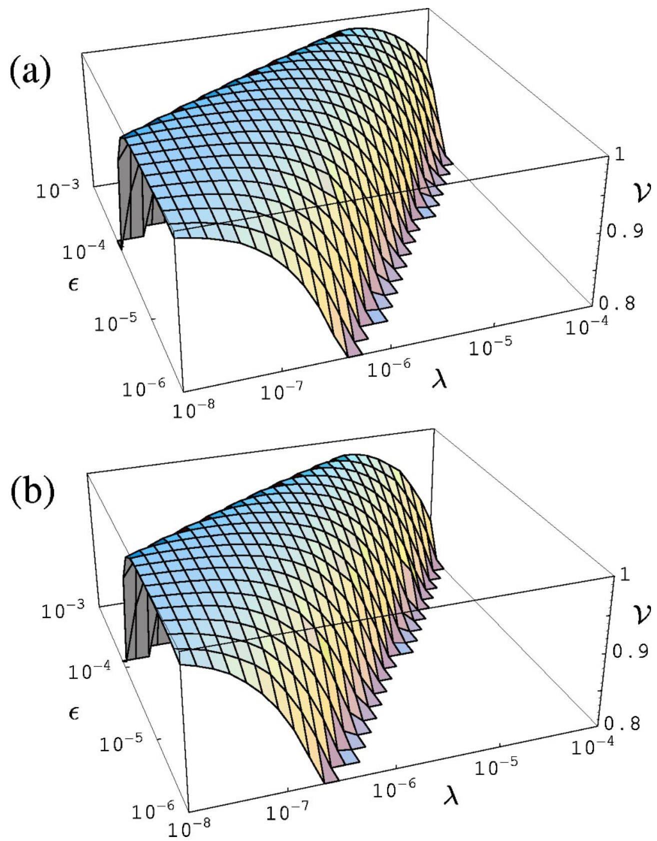

FIG. 1. The single-photon visibility with nonselective detectors as a function of the strengths of the SPDC sources: (a) the entangled ancilla gate, (b) the Knill gate. In both cases the input state was truncated at six-photon terms, and the maximization of the error was performed numerically.

qubit encoding as in Eq. (1). We will again get the three terms (7) possibly contributing to the error for $n=4$. The last term again leads to output terms, which all get postselected out in the coincidence basis. Unfortunately, the output terms produced by the second term do not get postselected out leading to inherent errors in the statistics we will observe. Notice, however, that all these terms will be proportional to λ^2, so again by making $\lambda \ll \epsilon$ we can scale these terms away with selective detectors at the expense of the count rate. With nonselective detectors there will again be an optimum λ, see Fig. 1(b), which is very similar to the previous gate.

V. CONCLUSION

We have examined three possible implementations for linear optics CNOT gates with a view to experimentally demonstrating their operation in the near future. If we consider demonstrating the gates with SPDC and coherent state sources and nonselective detectors, there is a clear advantage to the simplified KLM CNOT gate, where the inherent symmetries in the gate allow the use of two independent SPDC sources to supply the control, target, and ancillary photons, with errors from the use of non-Fock states making little contribution. The other two implementations suffer from errors introduced by the non-Fock state inputs, which cannot be postselected out. While the situation may be mitigated somewhat by using a weak SPDC source, this would occur at the expense of the count rate of valid events that may be collected from the gate. The conclusion we arrive at is that an experimental program focusing on the simplified KLM CNOT gate would then allow immediate characterization of the gate with current sources and detectors, with the operation of the gate in a nondestructive fashion becoming possible when single-photon sources and selective detectors become available.

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